

Thermal symmetry of the Markovian master equation

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The quantum Markovian master equation of the reduced dynamics of a harmonic oscillator coupled to a thermal reservoir is shown to possess thermal symmetry. This symmetry is revealed by a Bogoliubov transformation that can be represented by a hyperbolic rotation acting on the Liouville space of the reduced dynamics. The Liouville space is obtained as an extension of the Hilbert space through the introduction of tilde variables used in the thermofield dynamics formalism. The angle of rotation depends on the temperature of the reservoir, as well as the value of Planck's constant. This symmetry relates the thermal states of the system at any two temperatures. This includes absolute zero, at which purely quantum effects are revealed. The Caldeira-Leggett equation and the classical Fokker-Planck equation also possess thermal symmetry. We compare the thermal symmetry obtained from the Bogoliubov transformation in related fields and discuss the effects of the symmetry on the shape of a Gaussian wave packet.

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I. INTRODUCTION

In the dynamical formulation of a quantum system in the Hilbert space \mathcal{H} , a continuous symmetry is represented by a unitary transformation U that commutes with the Hamiltonian H [1]. When we extend the dynamics to the Liouville space $\mathcal{L} = \mathcal{H} \times \tilde{\mathcal{H}}$, the symmetry of H is translated into a symmetry of the Liouville operator, defined by $L = -i(H \times 1 - 1 \times H)$ for a system with unitary time evolution. However, the reverse is not true. The Liouville operator may contain a symmetry that has no counterpart in the Hamiltonian [2]. For example, consider a simple harmonic oscillator with Hamiltonian $H_0 = \omega_0 a^\dagger a$. This system possesses a non-degenerate discrete energy spectrum, E_n . Nevertheless, the energy spectrum of the corresponding Liouville operator $L_0 = -i(H_0 \times 1 - 1 \times H_0)$ depends on energy differences $E_m - E_n$, and therefore degeneracies may occur [3]. The occurrence of degeneracies in the spectrum of the Liouville operator is a signature for the presence of a symmetry in \mathcal{L} that has no counterpart in \mathcal{H} .

It often occurs that the system we are interested in is embedded in a larger system, for instance a thermal reservoir. The subsystem is an open interacting quantum system with a non-unitary (semigroup) time evolution. We obtain the reduced dynamics of the system by tracing out the degrees of freedom of the reservoir. When memory effects are further neglected, we obtain the quan-

tum Markovian master equation (MME) [4]. This formalism provides a framework for describing dissipative processes in various fields, such as quantum optics [5, 6], quantum Brownian motion [7, 8], quantum decoherence [9] and, more recently, quantum information theory [10]. The generator of the time evolution of the MME is a non-Hermitian collision operator with complex eigenvalues [11, 12]. For different thermal states of the system, degeneracies may occur in the eigenvalues of the collision (or Liouville) operator. These degeneracies indicate the presence of a *thermal* symmetry that is the main subject of this paper. In contrast to the discussion in Ref. [2], in which the authors considered an exact symmetry S of the Liouville operator in the sense of $[S, L] = 0$, we consider a symmetry U of the collision operator $K(b)$ up to a similarity transformation, $K(b') = UK(b)U^\dagger$, where b labels the different thermal states of the system.

A natural setting for formulating our discussion is provided by the thermofield dynamics (TFD) formalism [13]. In this formalism, the extension to the Liouville space is achieved by introducing the operator $A = a \times 1$ and the tilde conjugate operator $\tilde{A}^\dagger = 1 \times a$ for the annihilation operator a of the harmonic oscillator. A Bogoliubov transformation, which can be represented by a hyperbolic rotation, can then be defined to act on the pair of operators (A, \tilde{A}^\dagger) and its Hermitian conjugate pair [14]. Thermal effects of the reservoir enter our analysis through a temperature dependent angle in the hyperbolic rotation. As a result, the rotation connects different thermal states of the system; in particular, each temperature is related to the zero temperature that contains purely quantum effects (without thermal effects).

A thermal symmetry may also exist in a classical system, though it occurs in a different form. This is true

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for the classical Fokker-Planck equation [15] where the thermal symmetry results from a scale transformation on the phase space variables. We find that the Caldeira-Leggett equation [16] possesses a thermal symmetry as well. However, when we consider the thermal symmetry of MMEs derived under more general conditions [17–19], the thermal symmetry is no longer present.

Since the Bogoliubov transformation preserves the canonical commutation relations between the position and momentum operators, we can introduce a symplectic structure into the theory. The thermal symmetry can then be formulated as an element of the $\text{Sp}(2n, \mathbb{R})$ group acting on the phase space operators, which is often employed in the studies of “Gaussons,” or Gaussian pure states [20]. We also discuss how the generator of the thermal symmetry is related to the Lie algebra of the $\text{SU}(1, 1)$ group utilized in quantum optics [21] and compare the thermal symmetry to other canonical transformation discussed in the literature [22].

Since the thermal symmetry connects different thermal states of the system, it also dictates the change in shape of the density function in the coordinate space as a function of the temperature. It is known that a density function with Gaussian profile experiences simultaneous stretch and contraction along the diagonal and off-diagonal directions separately in the coordinate space as a result of a change in temperature [23]. We will show the connection between these phenomena through the thermal symmetry. We conclude with a discussion of our results.

II. THERMAL SYMMETRY IN MARKOVIAN MASTER EQUATION (MME)

Consider a harmonic oscillator interacting with a thermal reservoir. When memory effects can be neglected, e.g. in the weak coupling limit, the time evolution of the reduced density operator $\hat{\rho}$ of the harmonic oscillator is governed by a Markovian master equation (MME), $\partial\hat{\rho}/\partial t = K\hat{\rho}$. The time evolution operator is [5, 24]

$$K\hat{\rho} = (K_0 + K_d)\hat{\rho}, \quad (1)$$

where

$$K_0\hat{\rho} = -i\omega_0[a^\dagger a, \hat{\rho}] \quad (2)$$

$$K_d\hat{\rho} = c_1(2a\hat{\rho}a^\dagger - a^\dagger a\hat{\rho} - \hat{\rho}a^\dagger a) + c_2(2a^\dagger\hat{\rho}a - aa^\dagger\hat{\rho} - \hat{\rho}aa^\dagger), \quad (3)$$

in which we have made the rotating-wave approximation for the interaction [25] and a and a^\dagger satisfy the usual commutation relation. All the dissipative effects of the dynamics are contained in K_d . This MME generates a completely positive dynamical semigroup on the reduced dynamics [4].

In Eq. (2), ω_0 is the natural frequency of the harmonic

oscillator and c_1, c_2 in Eq. (3) are coefficients given by

$$c_1 = \frac{\gamma}{2}\left(b + \frac{1}{2}\right), \quad c_2 = \frac{\gamma}{2}\left(b - \frac{1}{2}\right). \quad (4)$$

In the weak coupling approximation, the damping constant γ is given by [24]

$$\gamma = \int_0^\infty d\omega 2\pi\lambda^2 |v(\omega)|^2 \delta(\omega - \omega_0) = 2\pi\lambda^2 |v(\omega_0)|^2, \quad (5)$$

where λ denotes the coupling constant of the interaction and $v(\omega)$ is a model dependent form factor. The delta function in Eq. (5) arises from the contribution of the resonance pole at ω_0 . The thermal properties of the reservoir are encoded in the parameter b . In the weak coupling limit, b can be determined from the following expression

$$\begin{aligned} \gamma b &= \int_0^\infty d\omega 2\pi\lambda^2 |v(\omega)|^2 \tilde{b}(\omega) \delta(\omega - \omega_0) \\ &= 2\pi\lambda^2 |v(\omega_0)|^2 \tilde{b}(\omega_0), \end{aligned} \quad (6)$$

where we obtain

$$b = \tilde{b}(\omega_0), \quad (7)$$

after using Eq. (5). Assuming that the reservoir is in thermal equilibrium, the parameter $\tilde{b}(\omega)$ is given by

$$\tilde{b}(\omega) = \frac{1}{2} + \frac{1}{e^{\hbar\omega\beta} - 1} = \frac{1}{2} \coth\left(\frac{1}{2}\hbar\omega\beta\right), \quad (8)$$

where $\beta = 1/k_B T$, k_B being Boltzmann’s constant and T being the temperature of the reservoir. The physical range of b that corresponds to $0 \leq T < \infty$ is $\frac{1}{2} \leq b < \infty$.

We will now formulate our discussion in terms of the Liouville space by following the superoperator formulation in TFD [13]. We introduce the (super-)operators

$$A \equiv a \times 1, \quad A^\dagger \equiv a^\dagger \times 1. \quad (9)$$

and their tilde conjugate operators [32]

$$\tilde{A} \equiv 1 \times a^\dagger, \quad \tilde{A}^\dagger \equiv 1 \times a. \quad (10)$$

These operators act on the density operator from the left according to, e.g., $A\tilde{A}\hat{\rho} = A(\hat{\rho}a^\dagger) = a\hat{\rho}a^\dagger$, etc. Hermitian and tilde conjugations are idempotent operations [33],

$$(A^\dagger)^\dagger = A, \quad (\tilde{A})^\dagger = A, \quad (11)$$

and they act on the product of two arbitrary operators X, Y as

$$(\eta XY)^\dagger = \eta^* Y^\dagger X^\dagger, \quad (\widetilde{\eta XY}) = \eta^* \tilde{X} \tilde{Y}, \quad (12)$$

where η is a complex number and $*$ denotes complex conjugate. The commutation relations of the operators are

$$[A, A^\dagger] = 1, \quad [\tilde{A}, \tilde{A}^\dagger] = 1, \quad (13)$$

while all other commutators vanish. We also have the useful relation

$$[A - \tilde{A}^\dagger, (A - \tilde{A}^\dagger)^\dagger] = 0. \quad (14)$$

In terms of these operators, the uncoupled and dissipative components of the collision (super-)operator become

$$\begin{aligned} K_0 &= -i\omega_0 (A^\dagger A - \tilde{A}^\dagger \tilde{A}) \\ &= -i\frac{\omega_0}{2} \left[(A + \tilde{A}^\dagger)^\dagger (A - \tilde{A}^\dagger) + (A + \tilde{A}^\dagger)(A - \tilde{A}^\dagger)^\dagger \right] \end{aligned} \quad (15)$$

and [34]

$$\begin{aligned} K_d &= \frac{\gamma}{2} \left[(A - \tilde{A}^\dagger)^\dagger (A + \tilde{A}^\dagger) - (A - \tilde{A}^\dagger)(A + \tilde{A}^\dagger)^\dagger \right] \\ &\quad - b\gamma (A - \tilde{A}^\dagger)^\dagger (A - \tilde{A}^\dagger), \end{aligned} \quad (16)$$

respectively.

Now consider the unitary operator

$$U(\theta) = e^{iG\theta}, \quad (17)$$

with the Hermitian generator

$$G \equiv i(A\tilde{A} - A^\dagger \tilde{A}^\dagger). \quad (18)$$

Under the action of U by conjugation, the A operators are transformed into

$$A' = UAU^\dagger = \cosh \theta A - \sinh \theta \tilde{A}^\dagger, \quad (19a)$$

$$\tilde{A}'^\dagger = U\tilde{A}^\dagger U^\dagger = -\sinh \theta A + \cosh \theta \tilde{A}^\dagger, \quad (19b)$$

with similar expressions for their Hermitian conjugate partners. Eq. (19) is a Bogoliubov transformation which preserves the commutation relations in Eq. (13). This transformation can be represented as a hyperbolic rotation on the operator pair (A, \tilde{A}^\dagger) by [14, 21]

$$\begin{pmatrix} A' \\ \tilde{A}'^\dagger \end{pmatrix} = R^{-1}(\theta) \begin{pmatrix} A \\ \tilde{A}^\dagger \end{pmatrix}, \quad (20)$$

where the rotation matrix is

$$R(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}. \quad (21)$$

The transformation on (A^\dagger, \tilde{A}) is obtained by taking the Hermitian conjugation of Eqs. (19) or (20). It should be stressed that $R(\theta)$ (or U) mixes the operators A and its tilde conjugate \tilde{A}^\dagger , and hence it is a rotation in the Liouville space that has no counterpart in the Hilbert space.

The effect of U on the operators K_0 and K_d can now be deduced by using the following linear combinations of Eqs. (19)

$$A' \mp \tilde{A}'^\dagger = e^{\pm\theta} (A \mp \tilde{A}^\dagger). \quad (22)$$

The uncoupled component of the dynamics K_0 has exact symmetry under U ,

$$\begin{aligned} K'_0(A, \tilde{A}^\dagger) &= UK_0(A, \tilde{A}^\dagger)U^\dagger = K_0(A', \tilde{A}'^\dagger) \\ &= K_0(A, \tilde{A}^\dagger), \end{aligned} \quad (23)$$

that is, $[U, K_0] = 0$. This symmetry in K_0 leads to the degeneracies in the spectrum of the operator $L_0 (= K_0)$ mentioned in the introduction. For the dissipative component of the dynamics K_d , the first line on the rhs of Eq. (16) has exact symmetry under U , but the second line, which is proportional to b , transforms up to a similarity transformation (or remains form invariant). Overall, we have

$$K'_d(A, \tilde{A}^\dagger; b) = K_d(A', \tilde{A}'^\dagger; b) = K_d(A, \tilde{A}^\dagger; b'), \quad (24)$$

where the transformed parameter b' is related to the angle of the rotation by

$$\theta = \frac{1}{2} \ln(b'/b). \quad (25)$$

For a reference thermal configuration b , the physical values of θ lie within the range $-(\ln 2b)/2 \leq \theta < \infty$, for $\frac{1}{2} \leq b' < \infty$. As a consequence of Eqs. (23) and (24), the hyperbolic rotation connects the full collision operators between reservoirs of different temperatures up to a similarity transformation

$$K(b') = U(\theta)K(b)U^\dagger(\theta). \quad (26)$$

We note that in Ref. [2], the term ‘‘Liouville symmetry’’ refers to the existence of a symmetry S that commutes with the Liouville operator L (or the collision operator K), i.e. $[L, S] = 0$. For our system, while the thermal symmetry $U(\theta)$ does not commute with K , it preserves the form of K up to a similarity transformation between different thermal states of the system (26).

A quantum statistical system is described by a density operator $\hat{\rho}$ defined on the Hilbert space \mathcal{H} . As we formulate the dynamics in the Liouville space \mathcal{L} , the system is now described by a density state vector $|\rho\rangle\rangle$, which corresponds to $\hat{\rho}$ [3, 21]. We have used the double ket notation to emphasize the fact that the state vector lies in \mathcal{L} . Under the thermal symmetry, the density state then transforms as

$$|\rho'(b)\rangle\rangle = U|\rho(b)\rangle\rangle = |\rho(b')\rangle\rangle \quad (27)$$

and the time evolution is now governed by $\partial|\rho(b')\rangle\rangle/\partial t = K(b')|\rho(b')\rangle\rangle$.

The existence of the thermal symmetry is related to the degeneracies in the eigenvalues of the collision operator $C(b) \equiv iK(b)$ in the following way. Consider the non-Hermitian eigenvalue problem of $C(b)$ [11, 12, 27],

$$C(b)|\rho_{mn}^\pm(b)\rangle\rangle = z_{mn}^\pm|\rho_{mn}^\pm(b)\rangle\rangle, \quad (28)$$

which has complex eigenvalues denoted by

$$z_{mn}^\pm = \pm n\omega_0 - i(m - n/2)\gamma, \quad m \geq n, \quad (29)$$

where $m, n = 0, 1, 2, 3, \dots$, and the decay rate of the (m, n) mode is $(m - n/2)\gamma$. Under the action of U , Eq. (28) is written

$$C(b')|\rho_{mn}^\pm(b')\rangle\rangle = z_{mn}^\pm|\rho_{mn}^\pm(b')\rangle\rangle. \quad (30)$$

Since the complex eigenvalue z_{mn}^\pm is independent of the thermal parameter b , degeneracy in z_{mn}^\pm can occur for different thermal states of the system. This degeneracy in the eigenvalues of the collision operator K is a consequence of the thermal symmetry.

The transformation U can be interpreted as generating a continuous change in the value of T (or \hbar). For a transformation of the system from a lower temperature to a higher one, we have $b' > b$ (equivalently, we can view this as a decrease in the value of \hbar) with a positive angle, $\theta > 0$. The system thus moves away from the quantum limit, while a transformation in the opposite direction with $\theta < 0$ drives the system closer to the quantum limit. In the following discussion, we shall fix the value of \hbar and interpret the transformation as generating a change in the temperature.

Under the U transformation, a system that is initially characterized by b is mapped into a different thermal state characterized by b' . Hence the term *thermal* symmetry. Moreover, all finite temperature states are related to the $T = 0$ ($b = \frac{1}{2}$) state through the transformation. Since the $T = 0$ state is a purely quantum state, this implies that for systems with a thermal symmetry, such as Eq. (1), thermal effects can be inferred from the effect of vacuum fluctuation by means of the thermal symmetry.

III. THERMAL SYMMETRY IN OTHER MMES

In the previous section, the thermal symmetry is presented in the superoperator form in Eq. (20), which clearly illustrates that the symmetry cannot be reduced to a symmetry in the Hilbert space. Other aspects of the symmetry can be discussed more conveniently in terms of the coordinate basis representation. For instance, the existence of a thermal symmetry in the classical Fokker-Planck equation becomes obvious when we compare it with the quantum MME in the Wigner's representation. The coordinate basis also proves to be more convenient when we extend our discussion to include other MMEs, where they acquire more simple forms.

We shall make use of the dimensionless coordinate $x \equiv (M\omega_0/\hbar)^{1/2}q$, where M is the mass of the harmonic oscillator and q is the ordinary spatial coordinate with dimensions of length. In the Liouville space, we then have the pair of coordinates (x, \tilde{x}) to describe the system, which will transform as a hyperbolic rotation similar to A , cf. Eq. (35) below. Writing the A operators in terms of the position and momentum operators of the harmonic oscillator, we have

$$A = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad A^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}), \quad (31)$$

where $\hat{p} = -i\partial/\partial\hat{x}$ is the dimensionless momentum. With similar expressions for the tilde operators, we find that the operators and their tilde counterparts separately satisfy the canonical commutation relations,

$$[\hat{x}, \hat{p}] = i, \quad [\tilde{x}, \tilde{p}] = i, \quad (32)$$

while all other commutators vanish.

With the definition

$$\rho(x, \tilde{x}) \equiv \langle\langle x; \tilde{x} | \rho \rangle\rangle \equiv \langle x | \hat{\rho} | \tilde{x} \rangle, \quad (33)$$

the A operators are represented by [12, 24]

$$\langle\langle x; \tilde{x} | A | \rho \rangle\rangle = \frac{1}{\sqrt{2}} \left(x + \frac{\partial}{\partial x} \right) \rho(x, \tilde{x}), \quad (34a)$$

$$\langle\langle x; \tilde{x} | A^\dagger | \rho \rangle\rangle = \frac{1}{\sqrt{2}} \left(x - \frac{\partial}{\partial x} \right) \rho(x, \tilde{x}), \quad (34b)$$

in the coordinate basis, with similar expressions for the tilde operators.

From Eq. (34), the pair of coordinates (x, \tilde{x}) transforms under the thermal symmetry just like the pair of (A, \tilde{A}^\dagger) operators,

$$\begin{pmatrix} x' \\ \tilde{x}' \end{pmatrix} = R^{-1}(\theta) \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}. \quad (35)$$

We also have the transformation equations

$$x' \pm \tilde{x}' = e^{\mp\theta} (x \pm \tilde{x}), \quad (36a)$$

$$\frac{\partial}{\partial x'} \pm \frac{\partial}{\partial \tilde{x}'} = e^{\pm\theta} \left(\frac{\partial}{\partial x} \pm \frac{\partial}{\partial \tilde{x}} \right), \quad (36b)$$

where $\exp\theta = \sqrt{b'/b}$. In Sec.IV A, we show that Eqs. (36) can also be generated by a symplectic transformation.

In terms of the coordinate basis, the generator of the unitary transformation is

$$G = i \left(x \frac{\partial}{\partial \tilde{x}} + \tilde{x} \frac{\partial}{\partial x} \right). \quad (37)$$

A. Thermal symmetry in Fokker-Planck equation

As a preparation to illustrate the existence of thermal symmetry in the classical Fokker-Planck equation [15], we first write the quantum MME in the coordinate basis before establishing its connection with the classical case.

In the coordinate basis, the collision operator takes the form

$$K(x, \tilde{x}; b) = K_0(x, \tilde{x}) + K_d(x, \tilde{x}; b), \quad (38)$$

where

$$K_0(x, \tilde{x}) = -i\frac{\omega_0}{2} \left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \tilde{x}^2} + x^2 - \tilde{x}^2 \right), \quad (39)$$

$$K_d(x, \tilde{x}; b) = \frac{\gamma}{4} \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{x}} \right) (x + \tilde{x}) - (x - \tilde{x}) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}} \right) \right] + b\frac{\gamma}{2} \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{x}} \right)^2 - (x - \tilde{x})^2 \right], \quad (40)$$

are written in a manifestly form invariant manner under the transformation in Eqs. (36).

As an intermediate step, we introduce a new pair of coordinates,

$$Q \equiv \frac{1}{2}(x + \tilde{x}), \quad r \equiv x - \tilde{x}, \quad (41)$$

where the (r, Q) coordinate system is related to (x, \tilde{x}) coordinate system by a counter clock-wise rotation of 45° . Eqs. (36) then imply that (r, Q) transform separately under the thermal symmetry as scale transformations

$$Q' = e^{-\theta} Q, \quad r' = e^\theta r, \quad (42)$$

where $\exp(\theta) = \sqrt{b'/b}$.

To go into the Wigner representation, another pair of coordinates, (P, Q) , where P is the Fourier conjugate variable of r , are used to describe the dynamics. Both coordinates transform with the same scaling factor under the thermal symmetry,

$$Q' = e^{-\theta} Q, \quad P' = e^{-\theta} P. \quad (43)$$

The collision operator in Wigner's representation is given by

$$K(Q, P; b) = -\omega_0 \left(P \frac{\partial}{\partial Q} - Q \frac{\partial}{\partial P} \right) + \frac{\gamma}{2} \left(\frac{\partial}{\partial Q} Q + \frac{\partial}{\partial P} P \right) + b\frac{\gamma}{2} \left(\frac{\partial^2}{\partial Q^2} + \frac{\partial^2}{\partial P^2} \right). \quad (44)$$

As is evident from Eq. (44), the thermal parameter b can be absorbed simultaneously into both coordinates, which is effectively a scaling transformation according to Eq. (43). The existence of the second derivative terms in both the Q and P coordinates implies that a Gaussian wave packet would tend to a more uniform distribution along both the Q and P -axis as the temperature increases [23].

In Wigner's representation, the collision operator has the same form as the classical Fokker-Planck equation [15] expressed in the classical phase space variables, except with b replaced by its classical analog

$$b_{cl} = k_B T / \hbar \omega_0. \quad (45)$$

Therefore, we conclude that thermal symmetry exists in the classical system as well. We can then run the argument in the reverse direction, starting from a given classical Fokker-Planck equation, we can transform the classical phase space variables corresponding to (P, Q) into the classical analog of the (x, \tilde{x}) coordinate system. A thermal hyperbolic symmetry can then be defined on these coordinates as well. Indeed, the coordinates (x, \tilde{x}) in the classical system correspond to the Bargmann-Segal representation [28], which is the classical analog of the coherent state representation in quantum theory [29].

Scale transformations of the form in Eq. (43) also apply in the coherent state representation of the MMEs [6], where the pair of coordinates $z = P - iQ$ and its complex conjugate z^* are used. A pair of "action-angle" variables (J, α) , where

$$J = \frac{1}{2}(P^2 + Q^2), \quad \alpha = \tan^{-1}(Q/P), \quad (46)$$

can also be defined for the quantum system in parallel to the classical action-angle variables (for example, see [12]). Under the thermal symmetry, J then scales as $\exp(-2\theta)J$, while α is left unaffected.

B. Thermal symmetry in Caldeira-Leggett equation

In the previous discussion, we have focused on the thermal symmetry of the MME in Eq. (1). We will now extend our consideration to other quantum MMEs. It is interesting to recognize that the Caldeira-Leggett (CL) equation [16] for the high temperature limit of a harmonic oscillator interacting with a thermal reservoir possesses this thermal symmetry as well. Indeed, the collision operator of the CL equation is given by

$$K_{cl}(x, \tilde{x}; b_{cl}) = K_0(x, \tilde{x}) - \gamma_1(x - \tilde{x}) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}} \right) - \gamma_1 b_{cl}(x - \tilde{x})^2, \quad (47)$$

where b_{cl} in Eq. (45) is the high temperature limit of b . For Eq. (47), the second line (with coefficient b_{cl}) transforms under the thermal symmetry with an overall multiplicative factor $\exp(2\theta) = b'_{cl}/b_{cl}$, whereas the other terms are invariant under the transformation. We therefore have the following transformation law for K_{cl} ,

$$K_{cl}(x, \tilde{x}; b_{cl}) \xrightarrow{U(\theta)} K_{cl}(x, \tilde{x}; b'_{cl}), \quad (48)$$

and the CL equation is form invariant under the thermal symmetry.

C. MMEs under more general conditions

In order to determine the thermal symmetry for an MME, it is crucial that we are able to define for $R(\theta)$

one single angle that is independent of the frequencies of the field ω . This global property of the transformation originates from the resonance effect expressed by the delta function $\delta(\omega - \omega_0)$ on the rhs of Eq. (6). Due to the delta function, b is restricted to the natural frequency of the reduced system ω_0 .

However, for MMEs derived under more general considerations (for finite temperature and without making the rotating-wave approximation) [17–19], the restriction of ω to ω_0 does not occur in some of the coefficients. Let us consider the Markovian limit of the collision operator for the MMEs in Refs. [17–19]. They have the general form

$$K_g(x, \tilde{x}; b_k) = K_0(x, \tilde{x}) - \gamma_2(x - \tilde{x}) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \tilde{x}} \right) - 2\gamma_2 b(x - \tilde{x})^2 + i\Gamma(x - \tilde{x}) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial \tilde{x}} \right), \quad (49)$$

where

$$\Gamma = P \int_0^\infty d\omega \frac{I(\omega)}{\omega^2 - \omega_0^2} \tilde{b}(\omega), \quad (50)$$

is a temperature dependent coefficient. The symbol P denotes the principal value of the integration and $I(\omega)$ denotes the spectral density of the field. Unlike the coefficient $\gamma_2 b$, cf. Eq. (6), all frequency modes contribute to Γ due to the absence of a delta function in the integrand.

The first 3 terms on the rhs of Eq. (49) have a structure similar to the CL equation and hence are form invariant under $U(\theta)$. However, the Γ -term destroys the form invariance of K_g for the following reason. To have the Γ -term remain form invariant, we require

$$\tilde{b}'(\omega) = e^{2\theta} \tilde{b}(\omega) \quad (51)$$

for all ω , or equivalently,

$$\frac{\coth(\hbar\omega/2k_B T')}{\coth(\hbar\omega_0/2k_B T)} = \frac{\coth(\hbar\omega_0/2k_B T')}{\coth(\hbar\omega_0/2k_B T)}, \quad (52)$$

after using Eqs. (8), (7) and (25). This equation has no non-trivial solutions for $T' \geq 0$ other than $T' = T$, for all $\omega > 0$.

We might think of introducing a transformation $U[\theta(\omega)]$ with a frequency dependent angle, where

$$\theta(\omega) \equiv \frac{1}{2} \ln[\tilde{b}'(\omega)/\tilde{b}(\omega)], \quad (53)$$

in order to restore the form invariance of K_g . However, since the field degrees of freedom have been traced out to obtain the MMEs, it is impossible to have all the $U[\theta(\omega)]$ (with different ω) acting on the (x, \tilde{x}) coordinates simultaneously. Therefore, this construction is not feasible.

One might also consider modification of Eqs. (36) in order to incorporate a more general transformation that might account for the form invariance of the Γ -term. However, the requirements for the invariance of the canonical commutation relations in Eqs. (32) along

with those for the form invariance of the first three terms on the rhs of Eq. (49) under the modified transformation are too restrictive for this modification to be possible. For instance, if we modify the transformation coefficient for the derivative operator in Eq. (36b) from $\exp(\pm\theta)$ to $\exp(\pm\phi)$, where ϕ is an angle to be determined by the form invariance requirement for Γ , while we keep Eq. (36a) unchanged, we find that the second term on the rhs of Eq. (49) no longer remains form invariant. Furthermore, the modified operators no longer satisfy the canonical commutation relations in Eqs. (32).

In conclusion, the thermal symmetry in Eq. (20) is applicable to MMEs of linear systems: (1) under the rotating-wave approximation with Eq. (1), (2) in the high temperature system with the CL equation (47), as well as (3) in the classical Fokker-Planck equation. The symmetry does not apply to MMEs under more general considerations, for example without the rotating-wave approximation and for finite temperature. An example is the HPZ equation (49) [18]. For non-linear systems, generalization to the Bogoliubov transformations in Eq. (21) should be envisaged.

IV. RELATIONS WITH OTHER TRANSFORMATIONS

In this section we discuss the relation of the thermal symmetry to other forms of transformations utilized in closely related fields.

A. Thermal symmetry as element of $\text{Sp}(2n, \mathbb{R})$

Since the thermal symmetry preserves the commutation relations of the A operators (13), it also preserves the canonical commutation relations for the position and momentum operators (32). We can write the commutation relations in a more compact form by introducing a column vector

$$\hat{X} = \begin{pmatrix} \hat{x} \\ \hat{\tilde{x}} \\ \hat{p} \\ \hat{\tilde{p}} \end{pmatrix}. \quad (54)$$

The commutation relations in Eq. (32) can then be expressed as

$$[\hat{X}_i, \hat{X}_j] = i\Omega_{ij}, \quad i, j = 1, 2, 3, 4, \quad (55)$$

where Ω is a non-singular, anti-symmetric matrix,

$$\Omega = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}, \quad (56)$$

in which 1_2 and 0_2 denote the 2×2 identity and null matrix respectively.

Eqs. (19) and their Hermitian conjugates can then be written as the action of an element of the symplectic group on \hat{X} ,

$$\hat{X}' = S^{-1} \hat{X}, \quad (57)$$

where $S \in \text{Sp}(4, \mathbb{R})$ (with $n = 2$) is given by

$$S = e^{\theta J} = \begin{pmatrix} R(\theta) & 0_2 \\ 0_2 & R^{-1}(\theta) \end{pmatrix}. \quad (58)$$

We have previously introduced the rotation matrix $R(\theta)$ in Eq. (21). The symplectic transformation S is generated by

$$J = \begin{pmatrix} \tilde{1}_2 & 0_2 \\ 0_2 & -\tilde{1}_2 \end{pmatrix}, \quad (59)$$

where

$$\tilde{1}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (60)$$

By construction, the symplectic group S preserves Ω

$$S\Omega S^T = \Omega, \quad (61)$$

as expected, where T denotes the transpose operation.

The unitary operator \bar{U} that corresponds to S acts on \hat{X} by conjugation such that

$$\bar{U}(S) \hat{X} \bar{U}^\dagger(S) \equiv \hat{X}' = S^{-1} \hat{X} \quad (62)$$

by Eq. (57). It can be expressed as $\bar{U}(S) = \exp[i\theta \bar{G}(J)]$, where $\bar{G}(J)$ is the Hermitian generator of \bar{U} that corresponds to J . It has the following expression [20],

$$\bar{G}(J) = \frac{1}{2} \hat{X}^T \Omega J \hat{X} = -(\hat{x} \tilde{\hat{p}} + \tilde{\hat{x}} \hat{p}) \quad (63)$$

which is identical to the generator of $U(\theta)$ in Eq. (37) when expressed in the coordinate basis. The transformation laws in Eqs. (36) then follow from Eq. (62).

Notice that we started with a one dimensional system, in which the pair of canonical operators (\hat{x}, \hat{p}) is available for describing the dynamics. We are able to introduce the symplectic transformation $\text{Sp}(4, \mathbb{R})$ on the four components vector \hat{X} only because the pair of tilde operators $(\tilde{\hat{x}}, \tilde{\hat{p}})$ are introduced when we go into the Liouville space.

B. Relation of the thermal symmetry with $\text{SU}(1, 1)$

There is a close relationship between $U(\theta)$ and the $\text{SU}(1, 1)$ group. Consider the Lie algebra $\text{su}(1, 1)$, which consists of 3 operators [21]

$$M_+ = A^\dagger \tilde{A}^\dagger, \quad M_- = A \tilde{A}, \quad (64a)$$

$$M_0 = \frac{1}{2}(A^\dagger A + \tilde{A}^\dagger \tilde{A} + 1). \quad (64b)$$

These operators satisfy the commutation relations

$$[M_-, M_+] = 2M_0, \quad [M_0, M_\pm] = M_\pm. \quad (65)$$

An arbitrary $\text{SU}(1, 1)$ transformation can be parameterized as

$$U(\vec{\theta}) \equiv e^{-\vec{\theta} \cdot \vec{M}} = e^{-\theta_+ M_+ - \theta_- M_- - \theta_0 M_0}, \quad (66)$$

with inverse $U^{-1}(\theta) = U^\dagger(\theta) = U(-\theta)$. By choosing the angles as follows,

$$\theta_+ = -\theta, \quad \theta_- = \theta, \quad \theta_0 = 0, \quad (67)$$

Eq.(66) reduces to Eq. (17), or $U(\theta) = \exp(iG\theta)$, where the Hermitian generator

$$G \equiv i(M_- - M_+) = i(A\tilde{A} - A^\dagger \tilde{A}^\dagger) \quad (68)$$

generates the Lorentz group $\text{SO}(1, 1)$ in the (x, \tilde{x}) -plane [30].

C. Comparison with other form of the Bogoliubov transformation

Let us compare the Bogoliubov (canonical) transformation discussed in Ref. [22] with the Bogoliubov transformation in Eqs. (19). It was shown in Ref. [22] that MME with the following dissipative components

$$\underline{K}_d = K_d + c_3 K_3 + c_3^* K_3^\dagger, \quad (69)$$

where

$$K_3 \hat{p} = 2a \hat{p} a - a a \hat{p} - \hat{p} a a, \quad (70)$$

can be recast in the standard form of the dissipative operator K_d in Eq. (1) by an $\text{SU}(1, 1)$ transformation on (a, a^\dagger) ,

$$\underline{a} = ua + va^\dagger. \quad (71)$$

Here, $u = \cosh \nu$ and $v = \exp(i\eta) \sinh \nu$ are coefficients related to c_1 , c_2 and c_3 . Even though this transformation reduces \underline{K}_d to the standard form K_d , the uncoupled component of the dynamics K_0 (which need not be Eq. (15)) may no longer retain its original form.

In terms of the A operators, Eq. (71) becomes

$$\underline{A} = uA + vA^\dagger, \quad (72)$$

which unlike the thermal hyperbolic rotation (20), does not mix the A with the tilde operators, \tilde{A} or \tilde{A}^\dagger . Hence this rotation *can* be reduced to a transformation in the Hilbert space. This shows that the $\text{SU}(1, 1)$ hyperbolic rotation symmetry of Eq. (71) is different from the thermal hyperbolic rotation symmetry considered in Section II.

V. OTHER ASPECTS OF THE THERMAL SYMMETRY

A. Changes in the shape of Gaussian wave packet with temperature

When the temperature of the reservoir changes, the profile of the density function $\rho(x, \tilde{x})$ changes in response [23]. If the temperature increases, the magnitude of the off-diagonal components of ρ decrease, a process called decoherence [9]. At the same time, the diagonal components of ρ stretch out to attain a more uniform distribution. The density function responds in the opposite way when the temperature decreases. With a Gaussian wave-packet as an example, we shall relate these behaviors of ρ through the thermal symmetry.

Consider the equilibrium solution of the collision operator $C(b)$ in Eq. (28) in terms of the (r, Q) coordinates (41) [6, 11, 12, 23],

$$\tilde{\rho}(Q, r; b) = \frac{1}{\sqrt{2\pi b}} \exp(-Q/2b - br^2/2), \quad (73)$$

which is a Gaussian wave packet in both the r and Q coordinates.

We now evaluate the averaged first and second moment of Q and r along the $r = 0$ and $Q = 0$ sections of $\tilde{\rho}$. Along the $r = 0$ section of $\tilde{\rho}$, the average value of the first moment of Q vanishes,

$$\langle Q \rangle_{r=0} \equiv \frac{\int_{-\infty}^{\infty} Q \tilde{\rho}(Q, 0, b) dQ}{\int_{-\infty}^{\infty} \tilde{\rho}(Q, 0, b) dQ} = 0, \quad (74)$$

where the notation $\langle \cdots \rangle_{r=0}$ means that we are evaluating the average along the $r = 0$ section of $\tilde{\rho}$. We also find that $\langle r \rangle_{Q=0} = 0$ [35]. The second moments can also be evaluated readily to give

$$\langle Q^2 \rangle_{r=0} = b, \quad \langle r^2 \rangle_{Q=0} = 1/b. \quad (75)$$

The dispersion of the wave packet $(\Delta Q, \Delta r)$ is defined as

$$\Delta Q = \sqrt{\langle Q^2 \rangle_{r=0} - \langle Q \rangle_{r=0}^2}, \quad (76)$$

with a corresponding expression for Δr defined for the $Q = 0$ section of $\tilde{\rho}$. Substituting Eqs. (74) and (75) into ΔQ and Δr , we obtain

$$\Delta Q = \sqrt{b}, \quad \Delta r = 1/\sqrt{b}. \quad (77)$$

When the temperature increases, we have $b' > b$ (or $\theta > 0$), and the density function is transformed into

$$\tilde{\rho}'(Q, r; b) = \tilde{\rho}(Q', r'; b) = \tilde{\rho}(Q, r; b'). \quad (78)$$

The dispersion of the transformed Gaussian wave packet can be evaluated to give

$$(\Delta Q)' = \sqrt{b'} = e^\theta \Delta Q, \quad (79a)$$

$$(\Delta r)' = \frac{1}{\sqrt{b'}} = e^{-\theta} \Delta r, \quad (79b)$$

Eqs. (79) show that when the temperature increases ($\theta > 0$), the Gaussian wave packet stretches away from the origin along the direction parallel to the Q -axis ($r = 0$ section), whereas it contracts towards the origin along the direction parallel to the r -axis ($Q = 0$ section). The extent of the stretch and contraction is governed by the angle θ , or equivalently, the amount of change in the temperature. The density function behaves in the opposite manner when the temperature decreases.

B. Disconnected regions in coordinate space

When acting on the coordinate basis, the hyperbolic rotation $R(\theta)$ preserves the bilinear form $x^2 - \tilde{x}^2$. Each point on the (x, \tilde{x}) -plane belongs to a family of curves $x^2 - \tilde{x}^2 = \text{const}$ generated by $R(\theta)$ and these curves never cross. Consequently, the (x, \tilde{x}) -plane can be divided into 3 disconnected regions, according to $|x| = |\tilde{x}|$, $|x| > |\tilde{x}|$ and $|x| < |\tilde{x}|$. The fact that these regions are disconnected shows that the hyperbolic rotation does not mix quantum correlations (off-diagonal component $\langle x|\rho|\tilde{x} \rangle$) with probability (diagonal component $\langle x|\rho|x \rangle$). For the one dimensional case considered in this paper, the $|x| > |\tilde{x}|$ ($|x| < |\tilde{x}|$) regions are further divided into 2 disconnected pieces, depending on the value of $\text{sgn}(x) = \pm$ ($\text{sgn}(\tilde{x}) = \pm$). For higher dimensions, besides an overall hyperbolic rotation dependent on the only available thermal parameter b , there may also be ordinary rotations among the coordinates $\mathbf{x} = (x_1, x_2, \dots)$ or $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \dots)$, without mixing between \mathbf{x} and $\tilde{\mathbf{x}}$.

In analogy with special relativity, we can equally parameterize the rotation matrix R in terms of a “velocity”-like parameter v . Indeed, let us define v by the relations [12, 14]

$$\cosh \theta \equiv 1/\sqrt{1-v^2}, \quad \sinh \theta \equiv v/\sqrt{1-v^2}. \quad (80)$$

We find that v takes the familiar form

$$v = \tanh \theta = \frac{b - \frac{1}{2}}{b + \frac{1}{2}} = \exp(-\hbar\omega_0\beta), \quad (81)$$

which is the relative probability of finding the frequency mode ω_0 at thermal equilibrium. It asymptotically approaches the “light”-speed $v \rightarrow 1$ (or $\theta \rightarrow \infty$), when $T \rightarrow \infty$, or $\hbar \rightarrow 0$ in the classical limit. Hence, the classical limit as well as the high temperature limit is a singular limit similar to the speed of light limit in relativistic theory.

VI. CONCLUSION

We have shown the existence of a thermal symmetry in the reduced dynamics of open quantum systems with non-unitary time evolution, for instance, in the MME with Eq. (1), in the Caldeira-Leggett equation, as well as in the classical Fokker-Planck equation. However, for

systems considered under more general conditions, such as finite temperature and in the absence of the rotating-wave approximation, the thermal symmetry is no longer present. The Hu-Paz-Zhang equation is an example of the MMEs for such systems.

The thermal symmetry is generated by a Bogoliubov transformation on the Liouville space of the reduced density operator. This symmetry gives rise to degeneracies in the complex energy eigenvalues of the dissipative collision operator. The presence of this symmetry is due to the formulation of the dynamics on the level of the Liouville space; it does not exist on the level of the Hilbert space.

As an important consequence of the symmetry, different thermal states of the system are connected, including absolute zero. Hence for systems observing the thermal symmetry, the effects of the thermal reservoir on the system can be inferred from the properties of the system at absolute zero, which does not contain thermal effects. From a different point of view, the effects of the symmetry can be regarded as changing the value of Planck's constant for the case of fixed temperature. The symmetry then establishes a connection between the quantum and classical limit of the system.

When represented in the coordinate basis, the thermal symmetry takes the form of a hyperbolic rotation on the dynamical variables of the reduced system. The angle of rotation depends on the amount of change in the temperature, and it governs the extent of stretch and contraction of the density function along the diagonal and off-diagonal directions in the position coordinate basis,

or the (x, \tilde{x}) -plane.

The thermal symmetry can be expressed in terms of a Bogoliubov transformation because the reduced system is a simple harmonic oscillator, i.e., a linear reduced system. Generalization of this symmetry to a nonlinear reduced system would require a generalization of the Bogoliubov transformation [31]. The thermal symmetry also enables the construction of a set of temperature dependent density states of a coupled oscillator. We will present the discussion elsewhere.

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- [1] E. P. Wigner, *Group Theory* (Academic Press, New York, 1959); A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1993).
 - [2] Alec Maassen van den Brink and A. M. Zagorskin, *Quantum Information Processing* **1**, 55 (2002); also available at arXiv:quant-ph/0112080.
 - [3] See, for example, T. Petrosky, I. Prigogine, *Adv. Chem. Phys.* **99**, 1 (1997). The spectrum is actually infinitely degenerate. It is also interesting to see how the degeneracies occurs in terms of the action-angle variables for a classical system presented in the monograph of Prigogine in Ref. [15].
 - [4] G. Lindblad, *Commun. Math. Phys.* **48**, 119 (1976); V. Gorini, A. Kossakowski and E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976); V. Gorini, A. Frigerio, M. Verri, A. Kossakowski and E. C. G. Sudarshan, *Rep. Math. Phys.* **13**, 149, (1978).
 - [5] W. Weidlich and F. Haake, *Z. Phys.* **185**, 30 (1965); M. Lax, *Phys. Rev.* **145**, 110 (1966); W. H. Louisell and J. H. Marburger, *IEEE J. Quant. Electron.* **3**, 348 (1967); R. J. Glauber, in *Quantum Optics*, Proceedings of the International School of Physics "Enrico Fermi", Course XLII (Academic, New York, 1969), p. 15.
 - [6] G. S. Agarwal, in *Progress in Optics XI*, edited by E. Wolf (North-Holland, 1973).
 - [7] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).
 - [8] C. W. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods With Applications to Quantum Optics*, 2nd Ed. (Springer Verlag, Berlin, 2000).
 - [9] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 1996); for a review, see W. H. Zurek, *Rev. Mod. Phys.* **75**, 715 (2003).
 - [10] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2000).
 - [11] H. J. Briegel and B. G. Englert, *Phys. Rev. A* **47**, 3311 (1993).
 - [12] B. A. Tay, Ph.D. Dissertation, University of Texas at Austin (2004).
 - [13] Y. Takahashi and H. Umezawa, *Coll. Phenom.* **2**, 55 (1975), reprinted in *Int. J. Mod. Phys. B* **10**, 1755 (1996); H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, 1982); N. P. Landsman and Ch. G. Weert, *Phys. Rep.* **145**, 141 (1987).
 - [14] A.E. Santana, J.M.C. Malbouisson, A.P.C. Malbouisson and F.C. Khanna, in *Non-Linear Dynamics and Fundamental Interactions*, edited by F. Khanna and D. Matra-

- sulov (Springer Netherlands, 2006).
- [15] The classical Fokker-Planck equation is a well-studied subject. See, for example, I. Prigogine, *Non-Equilibrium Statistical Mechanics* (Wiley Interscience, 1962); H. Risken, *The Fokker-Planck Equation - Methods of Solution and Applications* (Springer-Verlag Berlin Heidelberg, 1989).
 - [16] A. O. Caldeira and A. J. Leggett, *Physica A* **121**, 587 (1983).
 - [17] W. G. Unruh and W. H. Zurek, *Phys. Rev. D* **40**, 1071 (1989).
 - [18] B. L. Hu, J. P. Paz and Y. Zhang, *Phys. Rev. D* **45**, 2843 (1992).
 - [19] B. A. Tay and G. Ordóñez, *Phys. Rev. E* **73**, 016120, (2006).
 - [20] R. Simon, E.C.G. Sudarshan and N. Mukunda, *Phys. Rev. A* **36**, 3868 (1987); R. Simon, E.C.G. Sudarshan and N. Mukunda, *Phys. Rev. A* **37**, 3028 (1988).
 - [21] Masashi Ban, *Phys. Rev. A* **47**, 5093 (1993).
 - [22] A. K. Ekert and P. L. Knight, *Phys. Rev. A* **42**, 487 (1990).
 - [23] D. F. Walls and G. J. Milburn, *Phys. Rev. A* **31**, 2403 (1985).
 - [24] T. Petrosky and V. Barsegov, *Phys. Rev. E* **65**, 046102 (2002).
 - [25] C. Cohen-Tannouji, J. Dupont-Roc and G. Grynberg, *Atom-Photon Interactions, Basic Processes and Applications* (Wiley, New York, 1992).
 - [26] I. Prigogine, C. George, F. Henin and L. Rosenfeld, *Chemical Scripta* **4**, 5-32 (1973).
 - [27] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 1 (McGraw-Hill Book Company, Inc., 1953).
 - [28] V. Bargmann, *Commun. Pure Appl. Math.* **14**, 187 (1961); I.E. Segal, *Ill. J. Math.* **6**, 500 (1962); E. E. Segal and G. W. Mackey, *Mathematical Problems of Relativistic Physics* (American Mathematical Society, Providence, RI, 1963). For an application of the Bargmann-Segal representation in the classical Liouville space, see for example, T. Petrosky, G. Ordóñez, and I. Prigogine, *Phys. Rev. A* **68**, 022107 (2003).
 - [29] J.R. Klauder and E.C.G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968); J. R. Klauder and B. Skagerstam, *Coherent States: Application in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
 - [30] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications* (John Wiley & Sons, NY, 1974).
 - [31] T. Petrosky, unpublished.
 - [32] The idea of tilde conjugation \tilde{A} in TFD was introduced by Prigogine et. al. in non-equilibrium statistical theory [26], where it is referred to as adjoint conjugation A^a .
 - [33] We consider only the bosonic operators. If fermionic operators are included, two successive tilde conjugations will result in an extra phase σ , i.e. $(\tilde{\tilde{A}}) = \sigma A$, where $\sigma = 1$ for bosonic operators and -1 for fermionic operators.
 - [34] K_0 and K_d are invariant under tilde or adjoint conjugation [26], i.e. $\tilde{K}_0 = K_0$ and $\tilde{K}_d = K_d$.
 - [35] Note that $\langle \cdots \rangle_{Q=0}$ is not the usual trace average of an operator along the diagonal elements of the density matrix.